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# Gap Functions and Existence of Solutions to Generalized Vector Quasi-Equilibrium Problems*

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**Abstract.** This paper deals with generalized vector quasi-equilibrium problems. By virtue of a nonlinear scalarization function, the gap functions for two classes of generalized vector quasi-equilibrium problems are obtained. Then, from an existence theorem for a generalized quasi-equilibrium problem and a minimax inequality, existence theorems for two classes of generalized vector quasi-equilibrium problems are established.

**Key words:** generalized vector quasi-equilibrium problem, nonlinear scalarization function, minimax inequality

## 1. Introduction

Throughout this paper, let $X$, $V$ and $Z$ be three locally convex Hausdorff topological spaces, $E$ be a nonempty, compact and convex subset of $X$ and $D$ be a nonempty, compact subset of $Z$. We also assume that $C: X \to 2^V$ is a set-valued mapping such that $C(x)$ is a proper, closed and convex cone of $V$ with $\operatorname{int} C(x) \neq \emptyset$, for each $x \in X$. A vector-valued mapping $e$: $X \to V$ is said to be a continuous selection from $\operatorname{int} C(\cdot)$ if for any $x \in X$, $e(x) \in \operatorname{int} C(x)$. Let $K$: $E \to 2^E$ be a set-valued mapping with closed values, $Q$: $E \to 2^D$ and $F$: $E \times D \times E \to 2^V$ be two set-valued mapping.

Consider two classes of generalized vector quasi-equilibrium problems of finding $\bar{x} \in E$ and $\bar{z} \in Q(\bar{x})$ such that

$$\text{(GVQEP1)} \quad \bar{x} \in K(\bar{x}) \quad \text{and} \quad F(\bar{x}, \bar{z}, y) \subseteq V \setminus -\operatorname{int} C(\bar{x}), \quad \forall y \in K(\bar{x}),$$

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and of finding $\tilde{x} \in E$ and $\tilde{z} \in Q(\tilde{x})$ such that

$$\text{(GVQEP2)} \quad \tilde{x} \in K(\tilde{x}) \quad \text{and} \quad F(\tilde{x}, \tilde{z}, y) \subset -C(\tilde{x}), \quad \forall y \in K(\tilde{x}).$$

It is well known that the vector equilibrium problem provides a unified model of several classes of problems, for example, vector variational inequality problems, vector complementarity problems, vector optimization problems and vector saddle point problems. Many authors have intensively studied different types of vector equilibrium problems (see Refs. [1, 5, 8, 10, 11]).

The gap function approach is an important research method in variational inequality. One advantage of the introduction of gap functions in variational inequalities is that variational inequalities can be transformed into optimization problems. Then, powerful optimization solution methods and algorithms can be applied for finding solutions of variational inequalities.

Recently, some authors have investigated the gap functions for vector variational inequalities. In Ref. [4], Chen et al. introduced two set-valued functions as gap functions for two classes of vector variational inequality. In Ref. [11], Li et al. studied differential and sensitivity properties of the two classes of gap functions for vector variational inequalities and got an explicit expression of their contingent derivatives. In Ref. [14], Yang and Yao introduced two real-valued functions as gap functions for two classes of finite dimensional vector variational inequalities with set-valued mapping under Pareto partial order. From the computational point of view, the latter is more useful. In Ref. [13], Yang also investigated the gap function for a finite dimensional extended weak vector prevariational inequality. However, up to now, there is not any paper to investigate the gap functions for the problems (GVQEP1) and (GVQEP2). There are two main reasons: (1) Domination structure $C(x)$ of (GVQEP1) and (GVQEP2) is not fixed. It is a set-valued mapping of variable $x$ such that $C(x)$ is a proper, closed and convex cone of $V$ with $\operatorname{int} C(x) \neq \emptyset$, for each $x \in X$. (2) (GVQEP1) and (GVQEP2) shall be discussed in general locally convex Hausdorff topological space, but not in finite dimensional space. The methods used for generating the gap functions of finite dimensional vector variational inequalities cannot be directly used to generate the gap functions of (GVQEP1) and (GVQEP2).

In this paper, we shall first use the nonlinear scalarization function defined by Chen et al. [5] to introduce two real-valued functions. Then, we prove the two function are gap functions of (GVQEP1) and (GVQEP2), respectively. Finally, we obtain existence of solutions for (GVQEP1) and (GVQEP2) by using Theorem 3.1 in Ref. [11], Theorem 4 in Ref. [7] and the nonlinear scalarization function.

The rest of the paper is organized as follows. In Section 2, we recall some basic definitions, a minimax inequality theorem and an existence theorem for a generalized quasi-equilibrium problem. In Sections 3, we investigate the gap functions for (GVQEP1) and (GVQEP2), respectively. In Section 4, we show existence results for (GVQEP1) and (GVQEP2), respectively.

## 2. Preliminary Results

In this section, we shall recall the definitions of convex properties for set-valued mappings and of a nonlinear scalarization function and some results used in the following sections.

DEFINITION 2.1. *Let* $G: E \times D \to 2^V$ *be a set-valued mapping.*

(1) $G(x, \cdot)$ *is said to be* $C(x)$*-convex on* $D$ *for a fixed* $x \in E$ *if, for any* $y_1, y_2 \in D$ *and* $\lambda \in (0, 1)$,

$$\lambda G(x, y_1) + (1-\lambda) G(x, y_2) \subseteq G(x, \lambda y_1 + (1-\lambda) y_2) + C(x).$$

(2) $G(x, \cdot)$ *is said to be* $C(x)$*-properly quasi-convex on* $D$ *for a fixed* $x \in E$ *if, for any* $y_1, y_2 \in D$, $\lambda \in (0, 1)$ *and* $v \in G(x, \lambda y_1 + (1-\lambda) y_2)$ *there exists* $v_1 \in G(x, y_1)$ *or* $v_2 \in G(x, y_2)$ *such that*

$$v \in v_1 - C(x) \quad \text{or} \quad v \in v_2 - C(x).$$

(3) $G(x, \cdot)$ *is said to be* $C(x)$*-properly quasi-concave on* $D$ *for a fixed* $x \in E$ *if* $-G(x, \cdot)$ *is* $C(x)$*-properly quasi-convex on* $D$.

Let $e: X \to V$ be a vector-valued mapping and, for any $x \in X, e(x) \in \operatorname{int} C(x)$.

DEFINITION 2.2. *The nonlinear scalarization function* $\xi_e: X \times V \to \mathcal{R}$ *is defined by*

$$\xi_e(x, y) = \inf\{\lambda \in \mathcal{R} | y \in \lambda e(x) - C(x)\}.$$

THEOREM 2.1 (Theorem 2.1 [5]). *Let* $X$ *and* $V$ *be two locally convex Hausdorff topological vector spaces, and let* $C: X \to 2^V$ *be a set-valued mapping such that, for each* $x \in X$, $C(x)$ *is a proper, closed, convex cone in* $V$ *with* $\operatorname{int} C(x) \neq \emptyset$. *Furthermore, let* $e: X \to V$ *be a continuous selection from the set-valued map* $\operatorname{int} C(\cdot)$. *Define a set-valued mapping* $W: X \to 2^V$ *by* $W(x) = V \setminus \operatorname{int} C(x)$, *for* $x \in X$. *Then, it holds that*

(i) *If* $W(\cdot)$ *is upper semi-continuous on* $X$*, then* $\xi_e(\cdot,\cdot)$ *is upper semi-continuous on* $X\times V$*,*
(ii) *If* $C(\cdot)$ *is upper semi-continuous on* $X$*, then* $\xi_e(\cdot,\cdot)$ *is lower semi-continuous on* $X\times V$*.*

Note that for the detailed definitions of lower and upper semi-continuities and continuity of set-valued mappings, see Aubin and Ekeland [2, pp. 108–110].

THEOREM 2.2 (Theorem 4[7]). *Let* $E$ *and* $D$ *be nonempty convex, compact subsets of* $X$ *and* $V$*, respectively. If* $g\colon E\times D\to\mathcal{R}$ *a lower semi-continuous function on* $E\times D$ *such that*

(i) *for any* $x\in E, g(x,\cdot)$ *is quasi-concave on* $D$*;*
(ii) *for any* $y\in D$, $g(\cdot,y)$ *is quasi-convex on* $E$*.*

*Then,*

$$\min_{x\in E}\sup_{y\in D} g(x,y)=\sup_{y\in D}\min_{x\in E} g(x,y).$$

THEOREM 2.3. *Suppose that the following conditions hold:*

(i) $K\colon E\to 2^E$ *is a continuous mapping with compact and convex values on* $E$*;*
(ii) $\psi\colon E\times E\to\mathcal{R}$ *is upper semi-continuous on* $E\times E$*;*
(iii) *For any* $x\in E, \psi(x,x)\geqslant 0$*;*
(iv) *For every fixed* $x\in E$, $\psi(x,\cdot)$ *is quasi-convex on* $E$*.*

*Then, there exists an* $\bar{x}\in E$ *such that*

$$\bar{x}\in K(\bar{x}) \quad \textit{and} \quad \psi(\bar{x},y)\geqslant 0,\ \ \forall y\in K(\bar{x}).$$

*Proof.* From the proof process of Theorem 3.1 in Ref. [11], this result holds. □

## 3. Gap Functions for (GVQEP1) and (GVQEP2)

In this section, we shall obtain gap functions for (GVQEP1) and (GVQEP2) using the nonlinear scalarization function. Set

$$\tilde{K}=\{x\in E\mid x\in K(x)\}.$$

DEFINITION 3.1. *$g\colon \tilde{K} \to \mathcal{R}$ is said to be a gap function of (GVQEP1) or (GVQEP2) if*

(i) $g(x) \leqslant 0, \quad \forall x \in \tilde{K}$,
(ii) $g(\bar{x}) = 0$ *if and only if $\bar{x}$ is a solution of (GVQEP1) or (GVQEP2).*

Let $F\colon E \times D \times E \to 2^V$ be a set-valued mapping with compact values. Then, we may introduce respectively the mappings $\phi(x, z, y)\colon \tilde{K} \times D \times \tilde{K} \to \mathcal{R}$ and $\varphi(x, z, y)\colon \tilde{K} \times D \times \tilde{K} \to \mathcal{R}$ as follows:

$$\phi(x, z, y) = \min_{v \in F(x,z,y)} \xi_e(x, v),$$

and

$$\varphi(x, z, y) = \max_{v \in F(x,z,y)} \xi_e(x, v).$$

LEMMA 3.1. *Suppose that, for every fixed $x \in \tilde{K}$, $F(x, \cdot, \cdot)$ is a lower semi-continuous set-valued mapping with compact values on $D \times \tilde{K}$ and $K\colon E \to 2^E$ is a set-valued mapping with closed values on $\tilde{K}$. Then, for every fixed $x \in \tilde{K}$, $\max_{y \in K(x)}[-\phi(x, \cdot, y)]$ and $\max_{y \in K(x)} \varphi(x, \cdot, y)$ are all lower semi-continuous on $D$.*

*Proof.* Since, for every fixed $x \in \tilde{K}$, $F(x, \cdot, \cdot)$ is a lower semi-continuous set-valued mapping on $D \times \tilde{K}$, it follows from Proposition 19 in Section 3 of Chapter 1 [2] that $\phi(x, \cdot, \cdot)$ is upper semi-continuous on $D \times \tilde{K}$ for every fixed $x \in \tilde{K}$. Then, we have that $\max_{y \in K(x)}[-\phi(x, \cdot, y)]$ is lower semi-continuous on $D$. Similarly, from the lower semi-continuity of $F(x, \cdot, \cdot)$ for every fixed $x \in \tilde{K}$, we have that $\varphi(x, \cdot, \cdot)$ is lower semi-continuous on $D \times \tilde{K}$ for every fixed $x \in \tilde{K}$. Thus, we have that $\max_{y \in K(x)} \varphi(x, \cdot, y)$ is lower semi-continuous on $D$. □

Furthermore, suppose that $Q\colon E \to 2^D$ is a set-valued mappings with closed values. Then, from Lemma 3.1, we may introduce two real-valued functions $g_1$ and $g_2$ from $\tilde{K}$ to $\mathcal{R}$ as follows:

$$g_1(x) = -\min_{z \in Q(x)} \max_{y \in K(x)} [-\phi(x, z, y)], \tag{1}$$

and

$$g_2(x) = -\min_{z \in Q(x)} \max_{y \in K(x)} \varphi(x, z, y). \tag{2}$$

Now we shall prove that $g_1(x)$ and $g_2(x)$ are gap functions for (GVQEP1) and (GVQEP2), respectively.

THEOREM 3.1. *Suppose that, for every fixed* $x \in \tilde{K}$*,* $F(x, \cdot, \cdot)$ *is a lower semi-continuous set-valued mapping with compact values on* $D \times \tilde{K}$ *and* $K: E \to 2^E$ *and* $Q: E \to 2^D$ *are two set-valued mappings with closed values on* $\tilde{K}$*. If* $F(x, z, x) \cap -\partial C(x) \neq \emptyset, \forall x \in \tilde{K}$ *and* $z \in Q(x)$*, then* $g_1(x)$ *defined by* (1) *is a gap function of* $(GVQEP1)$*.*

*Proof.* Since $F(x, z, x) \cap -\partial C(x) \neq \emptyset, \forall x \in \tilde{K}$ and $z \in Q(x)$, there exists a $w_{xz} \in F(x, z, x)$ such that $w_{xz} \in -\partial C(x)$. It follows from Proposition 2.3(v) in Ref. [5] that

$$\xi_e(x, w_{xz}) = 0.$$

Then,

$$\min \xi_e(x, F(x, z, x)) \leqslant 0, \quad \forall x \in \tilde{K} \text{ and } z \in Q(x).$$

Naturally,

$$\max_{y \in K(x)} [-\min \xi_e(x, F(x, z, y))] \geqslant 0, \quad \forall x \in \tilde{K} \text{ and } z \in Q(x),$$

and

$$\min_{z \in Q(x)} \max_{y \in K(x)} [-\min \xi_e(x, F(x, z, y))] \geqslant 0, \quad \forall x \in \tilde{K}.$$

So,

$$g_1(x) \leqslant 0, \quad \forall x \in \tilde{K}. \tag{3}$$

Now suppose that there exists $\bar{x} \in \tilde{K}$ such that $g_1(\bar{x}) = 0$. It follows from Lemma 3.1 that there exists a $\bar{z} \in Q(\bar{x})$ such that

$$g_1(\bar{x}) = -\max_{y \in K(\bar{x})} [-\min \xi_e(\bar{x}, F(\bar{x}, \bar{z}, y))] = 0,$$

namely,

$$\min \xi_e(\bar{x}, F(\bar{x}, \bar{z}, y)) \geqslant 0, \quad \forall y \in K(\bar{x}).$$

From Proposition 2.3(iii) in Ref. [5], we have

$$v \notin -\operatorname{int} C(\bar{x}), \quad \forall v \in F(\bar{x}, \bar{z}, y) \text{ and } y \in K(\bar{x}),$$

namely,

$$F(\bar{x}, \bar{z}, y) \subset V \setminus -\operatorname{int} C(\bar{x}), \quad \forall y \in K(\bar{x}).$$

Thus, $(\bar{x}, \bar{z})$ is a solution of (GVQEP1).

Conversely, suppose that $(\bar{x}, \bar{z})$ is a solution of (GVQEP1). It follows from Proposition 2.3 in Ref. [5] that

$$\xi_e(\bar{x}, v) \geqslant 0, \quad \forall v \in F(\bar{x}, \bar{z}, y) \text{ and } y \in K(\bar{x}).$$

Then, we have

$$\max_{y \in K(\bar{x})} [-\min \xi_e(\bar{x}, F(\bar{x}, \bar{z}, y))] \leqslant 0,$$

$$\min_{z \in Q(\bar{x})} \max_{y \in K(\bar{x})} [-\min \xi_e(\bar{x}, F(\bar{x}, z, y))] \leqslant 0,$$

and

$$g_1(\bar{x}) \geqslant 0. \tag{4}$$

It follows from (3) and (4) that

$$g_1(\bar{x}) = 0.$$

Thus, the mapping $g_1(x)$ is a gap function of (GVQEP1). □

THEOREM 3.2. *Suppose that, for every fixed $x \in \tilde{K}$, $F(x, \cdot, \cdot)$ is a lower semi-continuous set-valued mapping with compact values on $D \times \tilde{K}$ and $K: E \to 2^E$ and $Q: E \to 2^D$ are two set-valued mappings with closed values on $\tilde{K}$. If $F(x, z, x) \cap -\partial C(x) \neq \emptyset, \forall x \in \tilde{K}$ and $z \in Q(x)$, then $g_2(x)$ defined by (2) is a gap function of (GVQEP2).*

*Proof.* Similar to the proof of Theorem 3.1, by $F(x, z, x) \cap -\partial C(x) \neq \emptyset$, $\forall x \in \tilde{K}$ and $z \in Q(x)$, there exists a $w_{xz} \in F(x, z, x)$ such that

$$\xi_e(x, w_{xz}) = 0.$$

Then,

$$\max \xi_e(x, F(x, z, x)) \geqslant 0, \quad \forall x \in \tilde{K} \text{ and } z \in Q(x),$$

and

$$g_2(x) = -\min_{z \in Q(x)} \max_{y \in K(x)} \max \xi_e(x, F(x, z, y)) \leqslant 0, \quad \forall x \in \tilde{K}. \tag{5}$$

Suppose that there exists $\bar{x} \in \tilde{K}$ such that $g_2(\bar{x}) = 0$. It follows from Lemma 3.1 that there exists a $\bar{z} \in Q(\bar{x})$ such that

$$g_2(\bar{x}) = -\max_{y \in K(\bar{x})} \max \xi_e(\bar{x}, F(\bar{x}, \bar{z}, y)) = 0,$$

namely,

$$\max \xi_e(\bar{x}, F(\bar{x}, \bar{z}, y)) \leqslant 0, \quad \forall y \in K(\bar{x}).$$

From Proposition 2.3(ii) in Ref. [5], we have

$$F(\bar{x}, \bar{z}, y) \subset -C(\bar{x}), \quad \forall y \in K(\bar{x}).$$

Thus, $(\bar{x}, \bar{z})$ is a solution of (GVQEP2).

Conversely, suppose that $(\bar{x}, \bar{z})$ is a solution of (GVQEP2). It follows from Proposition 2.3(ii) in Ref. [5] that

$$\xi_e(\bar{x}, v) \leqslant 0, \quad \forall v \in F(\bar{x}, \bar{z}, y) \text{ and } y \in K(\bar{x}).$$

Then, we have

$$g_2(\bar{x}) \geqslant 0. \tag{6}$$

It follows from (5) and (6) that

$$g_2(\bar{x}) = 0.$$

Thus, the mapping $g_2(x)$ is a gap function of (GVQEP2). □

*Remark* 3.1. In the paper [14], Yang and Yao investigated the gap function of the vector variational inequality (VVI) with a set-valued mapping $T$, which consists of finding $\bar{x} \in K$ and $\bar{t} \in T(\bar{x})$ such that

$$\langle \bar{t}, y - \bar{x} \rangle \notin -\operatorname{int} \mathcal{R}^l_+, \quad \forall y \in K, \tag{7}$$

where $T: X \to 2^{L(X, \mathcal{R}^l)}$ is a set-valued mapping with a compact set $T(x)$ for each $x$ and $K \subset X$ is a compact set.

Suppose that $C(x) \equiv \mathcal{R}^l_+$, $K(x) \equiv K, \forall x \in E$ and $Z = L(X, \mathcal{R}^l)$. Let $Q(x) \equiv T(x), \forall x \in E$ and $F(x, z, y) = \langle z, y - x \rangle$, for any $x, y \in E$ and $z \in D$. Then, (GVQEP1) reduces the (VVI) with a set-valued mapping (7). Naturally, all assumptions of Theorem 3.1 are satisfied. We have that $\tilde{K} = K$,

$$\phi(x, z, y) = \xi_e(x, \langle z, y - x \rangle),$$

and

$$g_1(x) = \max_{z \in T(x)} \min_{y \in K} \xi_e(x, \langle z, y - x \rangle).$$

Take $e=(1,\dots,1)^T \in \operatorname{int}\mathcal{R}^l_+$. It follows from [3] that

$$\xi_e(v)=\min_{1\leqslant i\leqslant l} v_i.$$

We have

$$g_1(x)=\max_{z\in T(x)}\min_{y\in K}\min_{1\leqslant i\leqslant l}(\langle z, y-x\rangle_i),$$

where $\langle z, y-x\rangle_i$ is the $i$th component of $\langle z, y-x\rangle$. Then $g_1(x)$ is equal to the gap function $g(x)$ introduced by Yang and Yao [14]. Thus, the gap function $g_1(x)$ is a generalization of one in Ref. [14].

## 4. Existences of Solutions for (GVQEP1) and (GVQEP2)

In this section, we shall use Theorems 2.2 and 2.3 and the nonlinear scalarization function to prove the existences of solutions for (GVQEP1) and (GVQEP2).

THEOREM 4.1. *Suppose that the following conditions hold:*

(i) $W(\cdot)=V\backslash\operatorname{int}C(\cdot)$ *is upper semi-continuous on* $X$ *and* $\operatorname{int}C(\cdot)$ *has a continuous selection* $e(\cdot)$*;*
(ii) $K: E\to 2^E$ *is continuous on* $E$ *and* $Q: E\to 2^D$ *is upper semi-continuous on* $E$*. For every* $x\in E$, $K(x)$ *and* $Q(x)$ *are compact and convex sets in* $X$ *and* $Z$*, respectively;*
(iii) $F: E\times D\times E\to 2^V$ *is a lower semi-continuous mapping with compact values on* $E\times D\times E$*;*
(iv) *For any* $x\in E$*, there exists a* $z_x\in Q(x)$, $F(x,z_x,x)\subseteq V\backslash -\operatorname{int}C(x)$*;*
(v) *For every fixed* $x\in E$ *and* $z\in Q(x)$, $F(x,z,\cdot)$ *is* $C(x)$*-convex;*
(vi) *For every fixed* $x\in E$ *and* $y\in K(x)$, $F(x,\cdot,y)$ *is* $C(x)$*-properly quasi-concave;*

*Then, there exist an* $\bar{x}\in E$ *and a* $\bar{z}\in Q(\bar{x})$ *such that*

$$\bar{x}\in K(\bar{x}) \qquad \text{and} \qquad F(\bar{x},\bar{z},y)\subseteq V\backslash -\operatorname{int}C(\bar{x}), \quad \forall y\in K(\bar{x}). \tag{8}$$

*Proof.* Suppose that

$$\psi(x,y)=\max_{z\in Q(x)}\left\{\min \xi_e(x, F(x,z,y))\right\}.$$

Now we prove that $\psi(x,y)$ satisfies the conditions of Theorem 2.3.

(1) $\psi: E\times E\to\mathcal{R}$ is upper semi-continuous on $E\times E$.

By assumption (i) and Theorem 2.1(i), we have that $\xi_e(x, v)$ is upper semi-continuous on $X \times V$. Naturally, if we consider the $\xi_e(x, v)$ as a function of the variable $(x, z, y, v)$, $\xi_e$ is also upper semi-continuous on $X \times Z \times X \times V$. It follows from the assumption (iii) and Proposition 19 in Section 1 of Chapter 3 [2] that $\min \cup_{v \in F(x,z,y)} \xi_e(x, v)$ is upper semi-continuous on $E \times D \times E$. Thus, from the assumption (i) and Proposition 20 in Section 1 of Chapter 3 [2], we have that $\psi(x, y)$ is upper semi-continuous on $E \times E$.

(2) For every fixed $x \in E$, $\psi(x, \cdot)$ is convex on $E$.
Since $F(x, z, y)$ is a compact set for any $y \in E$ and $z \in D$ and $\xi_e(x, \cdot)$ is continuous on $V$, $\xi_e(x, F(x, z, y))$ is a compact set for any $y \in E$. Suppose that $y_1, y_2 \in E$ and $\lambda \in (0, 1)$. Then, there exists a $v_{1z} \in F(x, z, y_1)$ and $v_{2z} \in F(x, z, y_2)$ such that

$$\xi_e(x, v_{1z}) = \min \xi_e(x, F(x, z, y_1)),$$

and

$$\xi_e(x, v_{2z}) = \min \xi_e(x, F(x, z, y_2)).$$

From the $C(x)$-convexity of $F(x, z, \cdot)$, there exist $v_z \in F(x, z, \lambda y_1 + (1 - \lambda)y_2)$ and $c_z \in C(x)$ such that

$$\lambda v_{1z} + (1 - \lambda)v_{2z} = v_z + c_z.$$

It follows from Ref. [5] that

$$\begin{aligned}\lambda\xi_e(x, v_{1z}) + (1 - \lambda)\xi_e(x, v_{2z}) &\geqslant \xi_e(x, \lambda v_{1z} + (1 - \lambda)v_{2z}) \\ &\geqslant \xi_e(x, v_z) \\ &\geqslant \min \xi_e(x, F(x, z, \lambda y_1 + (1 - \lambda)y_2)),\end{aligned}$$

namely,

$$\begin{aligned}&\lambda \min \xi_e(x, F(x, z, y_1)) + (1 - \lambda) \min \xi_e(x, F(x, z, y_2)) \\ &\quad \geqslant \min \xi_e(x, F(x, z, \lambda y_1 + (1 - \lambda)y_2)).\end{aligned}$$

Thus, $\psi(x, \cdot)$ is convex on $E$.

(3) For any $x \in E, \psi(x, x) \geqslant 0$.
By the assumption (iv) and Proposition 2.3(iii) in Ref. [5], we have that, for any $x \in E$, there exists $z_x \in Q(x)$ such that

$$\xi_e(x, v) \geqslant 0, \quad \forall v \in F(x, z_x, x).$$

Thus,

$$\min \xi_e(x, F(x, z_x, x)) \geqslant 0,$$

and

$$\psi(x, x) = \max_{z \in Q(x)} \min \xi_e(x, F(x, z, x)) \geqslant 0, \quad \forall x \in E.$$

So, by Theorem 2.3, there exists $\bar{x} \in E$ such that

$$\bar{x} \in K(\bar{x}) \qquad \text{and } \psi(\bar{x}, y) \geqslant 0, \quad \forall y \in K(\bar{x}). \tag{9}$$

Then, we have

$$\min_{y \in K(\bar{x})} \psi(\bar{x}, y) \geqslant 0,$$

namely,

$$\inf_{y \in K(\bar{x})} \max_{z \in Q(\bar{x})} \big(\min \xi_e(\bar{x}, F(\bar{x}, z, y))\big) \geqslant 0. \tag{10}$$

Now we prove that the function $-\min \xi_e(\bar{x}, F(\bar{x}, \cdot, \cdot))$ satisfies assumptions of Theorem 2.2.

(a) It follows from the proof process of the previous (1) that $-\min \xi_e(\bar{x}, F(\bar{x}, \cdot, \cdot))$ is lower semi-continuous on $Q(\bar{x}) \times K(\bar{x})$.
(b) From the proof process of the previous (2), we have that $-\min \xi_e(\bar{x}, F(\bar{x}, z, \cdot))$ is concave on $K(\bar{x})$ for every $z \in Q(\bar{x})$.
(c) For every fixed $y \in K(\bar{x})$, we have that $-\min \xi_e(\bar{x}, F(\bar{x}, \cdot, y))$ is quasi-convex on $Q(\bar{x})$ for every $y \in K(\bar{x})$.

In fact, we only need to prove that $\min \xi_e(\bar{x}, F(\bar{x}, \cdot, y))$ is quasi-concave on $Q(\bar{x})$. Suppose that $z_1, z_2 \in E$ and $\lambda \in (0, 1)$. Then, there exists $v_z \in F(\bar{x}, \lambda z_1 + (1-\lambda) z_2, y)$ such that

$$\xi_e(\bar{x}, v_z) = \min \xi_e(\bar{x}, F(\bar{x}, \lambda z_1 + (1-\lambda) z_2), y).$$

It follows from the $C(\bar{x})$-properly quasi-concavity of $F(\bar{x}, \cdot, y)$ that there exists $v_{z_1} \in F(\bar{x}, z_1, y)$ or $v_{z_2} \in F(\bar{x}, z_2, y)$ such that

$$v_z \in v_{z_1} + C(\bar{x}) \quad \text{or} \quad v_z \in v_{z_2} + C(\bar{x}).$$

Then,

$$\xi_e(\bar{x}, v_z) \geqslant \xi_e(\bar{x}, v_{z_1}) \quad \text{or} \quad \xi_e(\bar{x}, v_z) \geqslant \xi_e(\bar{x}, v_{z_2}).$$

Thus, we have

$$\xi_e(\bar{x}, v_z) \geqslant \min \xi_e(\bar{x}, F(\bar{x}, z_1, y)) \quad \text{or} \quad \xi_e(\bar{x}, v_z) \geqslant \min \xi_e(\bar{x}, F(\bar{x}, z_2, y)),$$

namely, $\min \xi_e(\bar{x}, F(\bar{x}, \cdot, y))$ is quasi-concave on $Q(\bar{x})$.

So, by Theorem 2.2, we get

$$\sup_{y\in K(\bar{x})} \min_{z\in Q(\bar{x})} \big(-\min \xi_e(\bar{x}, F(\bar{x}, z, y))\big) = \min_{z\in Q(\bar{x})} \sup_{y\in K(\bar{x})} \big(-\min \xi_e(\bar{x}, F(\bar{x}, z, y))\big),$$

i.e.,

$$\inf_{y\in K(\bar{x})} \max_{z\in Q(\bar{x})} \big(\min \xi_e(\bar{x}, F(\bar{x}, z, y))\big) = \max_{z\in Q(\bar{x})} \inf_{y\in K(\bar{x})} \big(\min \xi_e(\bar{x}, F(\bar{x}, z, y))\big).$$

By (10), we have

$$\max_{z\in Q(\bar{x})} \inf_{y\in K(\bar{x})} \big(\min \xi_e(\bar{x}, F(\bar{x}, z, y))\big) \geqslant 0.$$

Since $\min \xi_e(\bar{x}, F(\bar{x}, \cdot, \cdot))$ is upper semi-continuous on $E \times D$, $\inf_{y\in K(\bar{x})}$ $(\min \xi_e(\bar{x}, F(\bar{x}, \cdot, y)))$ is upper semi-continuous on $D$. Then, there exists $\bar{z} \in Q(\bar{x})$ such that

$$\min \xi_e(\bar{x}, F(\bar{x}, \bar{z}, y)) \geqslant 0, \quad \forall y \in K(\bar{x}).$$

It follows from Proposition 2.3(iii) in Ref. [5] that

$$F(\bar{x}, \bar{z}, y) \subseteq V \backslash -\mathrm{int}\, C(\bar{x}), \quad \forall y \in K(\bar{x}). \tag{11}$$

Thus, by (9) and (11), the proof is complete. □

*Remark* 4.1. Suppose that $F(\cdot,\cdot,\cdot) = f(\cdot,\cdot,\cdot)$ is a vector-valued function. In Ref. [5], Chen et al. discussed the following generalized quasi-equilibrium problem (GQEP):

Find $\bar{x} \in K(\bar{x})$ and $\bar{z} \in Q(\bar{x})$ such that
$f(\bar{x}, \bar{z}, \bar{x}) - f(\bar{x}, \bar{z}, y) \notin \mathrm{int}\, C(\bar{x}), \quad \forall y \in K(\bar{x}).$

By using Fan–Glicksber–Kakutani fixed point theorem and nonlinear scalarization function $\xi$, they proved an existence theorem of solutions for (GQEP). In this paper, by the virtue of Ky Fan minimax inequality (Theorem 2.2), the existence theorem of solutions for generalized quasi-variational inequality (Theorem 2.3) and nonlinear scalarization function $\xi$, we obtain an existence theorem for (GVQEP1) (Theorem 4.1). Although the two papers use all nonlinear scalarization function $\xi$ to prove their existence theorems respectively, the conditions that the two existence theorems hold and proof methods are different.

Now we prove the existence of a solution for (GVQEP2).

THEOREM 4.2. *Suppose that the following conditions hold:*

(i) *$C: X \to 2^Z$ is upper semi-continuous on $X$ and* $\operatorname{int} C(\cdot)$ *has a continuous selection $e(\cdot)$;*
(ii) *$K: E \to 2^E$ is continuous on $E$ and $Q: E \to 2^D$ is upper semi-continuous on $E$. For every $x \in E$, $K(x)$ and $Q(x)$ are compact and convex sets in $X$ and $Z$, respectively;*
(iii) *$F: E \times D \times E \to 2^V$ is a lower semi-continuous mapping with compact values on $E \times D \times E$;*
(iv) *For any $x \in E$, there exists a $z_x \in Q(x)$, $F(x, z_x, x) \subseteq -C(x)$;*
(v) *For every fixed $x \in E$ and $z \in Q(x)$,* $\max \xi_e(x, F(x, z, \cdot))$ *is quasi-concave;*
(vi) *For every fixed $x \in E$ and $y \in K(x)$, $F(x, \cdot, y)$ is $C(x)$-properly quasi-convex;*

*Then, there exist an $x^* \in E$ and a $z^* \in Q(x^*)$ such that*

$$x^* \in K(x^*) \qquad \text{and } F(x^*, z^*, y) \subseteq -C(x^*), \quad \forall y \in K(x^*). \tag{12}$$

*Proof.* Suppose

$$\omega(x, y) = -\min_{z \in Q(x)} (\max \xi_e(x, F(x, z, y))).$$

Following the proof process for (1)–(3) of Theorem 4.1, we have that $\omega(x, y)$ satisfies all assumptions of Theorem 2.3. Then, there is an $x^* \in E$ such that

$$x^* \in K(x^*) \qquad \text{and } \omega(x^*, y) \geqslant 0, \quad \forall y \in K(x^*), \tag{13}$$

i.e.,

$$\sup_{y \in K(x^*)} \min_{z \in Q(x^*)} \big(\max \xi_e(x^*, F(x^*, z, y))\big) \leqslant 0. \tag{14}$$

Following the proof process for (a) and (c) of Theorem 4.1, we get that $\max \cup_{v \in F(x^*, z, y)} \xi_e(x^*, v)$ satisfies the assumptions of Theorem 2.2. It follows from (14) that

$$\min_{z \in Q(x^*)} \sup_{y \in K(x^*)} \big(\max \xi_e(x^*, F(x^*, z, y))\big) \leqslant 0.$$

By the lower semi-continuity of $\max\xi_e(x^*, F(x^*, \cdot, \cdot))$, $\sup_{y\in K(x^*)}(\max\xi_e(x^*, F(x^*, \cdot, y)))$ is lower semi-continuous on $D$. Then, by (13) and Proposition 2.3(ii) in Ref. [5], there exists $z^* \in Q(x^*)$ such that

$$x^* \in K(x^*) \qquad \text{and } z \in -C(x^*),\ \forall y \in K(x^*) \quad \text{and } z \in F(x^*, z^*, y).$$

Thus, (12) holds and this completes the proof. □

*Remark* 4.2. If, for each fixed $x \in E$ and $y \in K(x)$, $-F(x, \cdot, y)$ is $C(x)$-properly quasi-convex introduced in Ref. [9] on $E$, then, $\max\xi_e(x, F(x, \cdot, y))$ is quasi-concave on $E$. However, the converse relation may not hold.

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